Heterokedasticity and Autocorrelation

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Gaus-Markov Regression Model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + \varepsilon_i \qquad i = 1, 2, \ldots, n$$

or

$$y_i = \mathbf{x}_i' \boldsymbol{\beta} + \varepsilon_i \qquad i = 1, 2, \ldots, n$$

or

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

$\mathrm{E}(\varepsilon_i) = 0, \ i = 1, 2, \ldots, n$

$\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ and $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ are independent

$\mathrm{var}(\varepsilon_i) = \sigma^2, \ i = 1, 2, \ldots, n$

$\mathrm{cov}(\varepsilon_i, \varepsilon_j) = 0, \ i, j = 1, 2, \ldots, n, \ i \neq j$

or

$$\mathrm{E}(\boldsymbol{\varepsilon} | \mathbf{X}) = \mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0}$$

$$\mathrm{cov}(\boldsymbol{\varepsilon} | \mathbf{X}) = \mathrm{cov}(\boldsymbol{\varepsilon}) = \sigma^2 \mathbf{I}$$

Gaus-Markov Regression Model

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \mathbf{X} = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1p} \\ 1 & x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & x_{np} \end{bmatrix}, \ \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}, \ \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix}$$

$$\mathrm{E}\left(\boldsymbol{\varepsilon} \middle| \mathbf{X}\right) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathrm{cov}\left(\boldsymbol{\varepsilon} \middle| \mathbf{X}\right) = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}$$

Gaus-Markov Regression Model

OLS estimator

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y} \text{ with } \operatorname{cov}\left(\hat{\boldsymbol{\beta}}\right) = \sigma^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

is the best linear unbiased estimator of $\boldsymbol{\beta}$.

- Its expectation values are equal to its parameters
- It is a linear function of the random variable $\mathbf{y}$.
- It has minimum variances among other unbiased estimator

Heteros-kedasticity and autocorrelation

$$\operatorname{cov}\left(\boldsymbol{\varepsilon}\middle|\mathbf{X}\right)=\begin{bmatrix}\sigma_1^2 & \sigma_{21} & \cdots & \sigma_{n1}\\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{n2}\\ \vdots & \vdots & \ddots & \vdots\\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2\end{bmatrix}=\sigma^2\boldsymbol{\Psi};\ \boldsymbol{\Psi}\neq\mathbf{I}$$

- Heteroskedasticity: at least there is one variance being different from the rest.
- Autocorrelation: at least there is one covariance being not zero.

Heteros-kedasticity and autocorrelation consequence on OLS estimator

$$\operatorname{var}\left(\hat{\boldsymbol{\beta}}\middle|\mathbf{X}\right) = \operatorname{var}\left\{\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\middle|\mathbf{X}\right\}$$

$$= \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\operatorname{var}\left(\boldsymbol{\varepsilon}\middle|\mathbf{X}\right)\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

$$= \sigma^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Psi}\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

$$\sigma^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Psi}\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1} = \sigma^2\left(\mathbf{X}'\mathbf{X}\right)^{-1} \Leftarrow \boldsymbol{\Psi} = \mathbf{I}$$

A wrong OLS estimator’s covariance matrix: misleading inferences, not longer best.

But still unbiased (proof it)

Remedies

- Alternative estimators
- Adjusting standard error

Alternative estimators

Assuming $\boldsymbol{\Psi}$ is positive definite, there is always a non-singular square matrix $\mathbf{P}$ which satisfies

$$\boldsymbol{\Psi}^{-1} = \mathbf{P}'\mathbf{P}$$

$$\boldsymbol{\Psi} = \left(\mathbf{P}'\mathbf{P}\right)^{-1} = \mathbf{P}^{-1}\left(\mathbf{P}'\right)^{-1}$$

$$\mathbf{P}\boldsymbol{\Psi}\mathbf{P}' = \mathbf{P}\mathbf{P}^{-1}\left(\mathbf{P}'\right)^{-1}\mathbf{P}' = \mathbf{I}$$

Consequently, the transformation of $\boldsymbol{\varepsilon}$ by $\mathbf{P}$ provides

$$\mathrm{E}\left(\mathbf{P}\boldsymbol{\varepsilon}\middle|\mathbf{X}\right) = \mathbf{P}\,\mathrm{E}\left(\boldsymbol{\varepsilon}\middle|\mathbf{X}\right) = \mathbf{0}$$

$$\mathrm{cov}\left(\mathbf{P}\boldsymbol{\varepsilon}\middle|\mathbf{X}\right) = \mathbf{P}\,\mathrm{cov}\left(\mathbf{P}\boldsymbol{\varepsilon}\middle|\mathbf{X}\right)\mathbf{P}' = \sigma^2\mathbf{P}\boldsymbol{\Psi}\mathbf{P}' = \sigma^2\mathbf{I}$$

satisfying Gauss-Markov condition.

Alternative estimators

Transforming the entire model by $\mathbf{P}$ we derive

$$\mathbf{Py} = \mathbf{PX\beta} + \mathbf{P\varepsilon} \text{ or } \mathbf{y}^* = \mathbf{X}^*\boldsymbol{\beta} + \boldsymbol{\varepsilon}^*$$

Applying OLS, we obtain the GLS estimator

$$\hat{\beta}_{GLS} = \left(\mathbf{X}^{*\prime}\mathbf{X}^*\right)^{-1}\mathbf{X}^{*\prime}\mathbf{y}^* = \left(\mathbf{X}'\boldsymbol{\Psi}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Psi}^{-1}\mathbf{y}$$

- with

$$\operatorname{cov}\left(\hat{\beta}_{GLS}\right) = \sigma^2\left(\mathbf{X}^{*\prime}\mathbf{X}^*\right)^{-1} = \sigma^2\left(\mathbf{X}'\boldsymbol{\Psi}^{-1}\mathbf{X}\right)^{-1}$$

$$\hat{\sigma}^2 = \frac{1}{n-p-1}\left(\mathbf{y}^* - \mathbf{X}^*\hat{\boldsymbol{\beta}}_{GLS}\right)'\left(\mathbf{y}^* - \mathbf{X}^*\hat{\boldsymbol{\beta}}_{GLS}\right)$$

$$= \frac{1}{n-p-1}\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}_{GLS}\right)'\boldsymbol{\Psi}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}_{GLS}\right)$$

If we replace $\boldsymbol{\Psi}$ by its estimate, we obtain the FGLS or EGLS estimator.

WLS (a special case of GLS for heteroskedasticity)

When

$$\sigma^2\boldsymbol{\Psi} = \sigma^2 \begin{bmatrix} h_1^2 & 0 & \cdots & 0 \\ 0 & h_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_n^2 \end{bmatrix} = \sigma^2 diag\left(h_i^2\right) \Rightarrow \mathbf{P} = diag\left(h_i^{-1}\right)$$

$$\mathbf{y}^* = \mathbf{X}^*\boldsymbol{\beta} + \boldsymbol{\varepsilon}^*$$

$$\frac{y_i}{h_i} = \left(\frac{\mathbf{x}_i'}{h_i}\right)\boldsymbol{\beta} + \frac{\varepsilon_i}{h_i}$$

$$\hat{\beta}_{GLS} = \left(\sum_{i=1}^{n} h_i^2 \mathbf{x}_i \mathbf{x}_i'\right)^{-1} \sum_{i=1}^{n} h_i^2 \mathbf{x}_i y_i$$

WLS in practice: multiplicative heteroscedasticity

We never know the weight for each individual hence by assuming

$$\sigma_i^2 = \sigma^2 \exp\left(\mathbf{z}_i'\boldsymbol{\alpha}\right)$$

We conduct the following procedure

- Calculate logarithm from squared OLS residuals
- Regress it a constant term and z, estimate alpha
- Calculate

$$\hat{h}_i = \sqrt{\exp\left(\mathbf{z}_i'\hat{\boldsymbol{\alpha}}\right)}$$

- Run OLS on

$$\frac{y_i}{\hat{h}_i} = \left(\frac{\mathbf{x}_i'}{\hat{h}_i}\right)\boldsymbol{\beta} + \frac{\varepsilon_i}{\hat{h}_i}$$

  to obtain EGLS estimate for beta.
- Calculate the standard error sigma square and standard error estimates accordingly

Heteroskedasticity-consistent (White's) standard error

- Heteroskedastic model

$$\operatorname{cov}\left(\boldsymbol{\varepsilon}\middle|\mathbf{X}\right)=\sigma^2\boldsymbol{\Psi}=\begin{bmatrix}\sigma_1^2 & 0 & \cdots & 0\\ 0 & \sigma_2^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \sigma_n^2\end{bmatrix}=diag\left(\sigma_i^2\right)$$

- The OLS estimator covariance matrix

$$\operatorname{cov}\left(\hat{\boldsymbol{\beta}}\middle|\mathbf{X}\right)=\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\operatorname{diag}\left(\sigma_i^2\right)\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

- Its consistent estimator

$$\hat{\operatorname{cov}}\left(\hat{\beta}\right)=\left(\mathbf{X}'\mathbf{X}\right)^{-1}\sum_{i=1}^{n}e_i^2\mathbf{x}_i\mathbf{x}_i'\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

$$\hat{\operatorname{cov}}\left(\hat{\boldsymbol{\beta}}\right)=\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\operatorname{diag}\left(e_i^2\right)\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$